The Taylor Series Expansion

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One of the most relevant series in Economics and Finance is the Taylor Series Expansion. In this white paper we will use the power series to build a Taylor Series Expansion. To assist us in this task we will use the following Hypothetical problem...

Our Hypothetical Problem

Use a Taylor Series Expansion to answer the following questions given this equation for $f(x)$:

$$f(x) = 100 - 2\,x + 4\,x^2 - 3\,x^3 \tag{1}$$

Question 1: What is the value of $f(x)$ when $x = 5$?

Question 2: What is the value of $f(x)$ when $x = -3$?

Building The Taylor Series Expansion

A Taylor series is a representation of a function $f(x)$ as an infinite sum of terms that are calculated from the values of the function's derivatives at a single point, which we will call point a. If $f(x)$ is a polynomial then we can write that function as the following power series...

$$f(x) = \sum_{n=0}^{\infty} m_n\,(x-a)^n = m_0 + m_1(x-a) + m_2(x-a)^2 + m_3(x-a)^3 + ... + m_\infty(x-a)^\infty \tag{2}$$

To solve the equation above we need to determine the values of m_0, m_1, m_2, etc. The general idea will be to process both sides of Equation (2) above and choose values of x so that only one unknown appears each time. With this plan in mind let's solve the equation...

Step 1: Set $x = a$ so that the terms m_1, m_2, m_3, etc. are removed from Equation (2) above. Once those terms are removed then we have the following equation...

$$f(x) = m_0 \text{ ...where... } x = a \tag{3}$$

Step 2: Take the derivative of Equation (2) above with respect to x...

$$\frac{\delta f(x)}{\delta x} = m_1 + 2\,m_2(x-a) + 3\,m_3(x-a)^2 + 4\,m_4(x-a)^3 + 5\,m_5(x-a)^4 + ... \tag{4}$$

Set $x = a$ so that the terms m_2, m_3, m_4, etc. are removed from Equation (4) above. Once those terms are removed then we have the following equation...

$$\frac{\delta f(x)}{\delta x} = f'(x) = m_1 \text{ ...where... } x = a \tag{5}$$

Step 3: Take the derivative of Equation (4) above with respect to x...

$$\frac{\delta^2 f(x)}{\delta x^2} = 2\,m_2 + 6\,m_3(x-a) + 12\,m_4(x-a)^2 + 20\,m_5(x-a)^3 + ... \tag{6}$$

Set $x = a$ so that the terms m_3, m_4, m_5, etc. are removed from Equation (6) above. Once those terms are removed then we have the following equation...

$$\frac{\delta^2 f(x)}{\delta x^2} = f''(x) = 2\,m_2 \ \text{...such that...}\ \frac{1}{2}\,f''(x) = m_2 \ \text{...where...}\ x = a \tag{7}$$

Step 4: Take the derivative of Equation (6) above with respect to x...

$$\frac{\delta^3 f(x)}{\delta x^3} = 6\,m_3 + 24\,m_4(x-a) + 60\,m_5(x-a)^2 + 120\,m_6(x-a)^3 + ... \tag{8}$$

Set $x = a$ so that the terms m_4, m_5, m_6, etc. are removed from Equation (8) above. Once those terms are removed then we have the following equation...

$$\frac{\delta^3 f(x)}{\delta x^3} = f'''(x) = 6\,m_3 \ \text{...such that...}\ \frac{1}{6}\,f'''(x) = m_3 \ \text{...where...}\ x = a \tag{9}$$

Step 5: Take the derivative of Equation (8) above with respect to x...

$$\frac{\delta^4 f(x)}{\delta x^4} = 24\,m_4 + 120\,m_5(x-a)^1 + 360\,m_6(x-a)^2 + 840\,m_7(x-a)^3 + ... \tag{10}$$

Set $x = a$ so that the terms m_5, m_6, m_7, etc. are removed from Equation (10) above. Once those terms are removed then we have the following equation...

$$\frac{\delta^4 f(x)}{\delta x^4} = f''''(x) = 24\,m_4 \ \text{...such that...}\ \frac{1}{24}\,f''''(x) = m_4 \ \text{...where...}\ x = a \tag{11}$$

Using Equations (3), (5), (7), (9) and (11) above we can rewrite Equation (2) above as...

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}\,f''(a)(x-a)^2 + \frac{1}{6}\,f'''(a)(x-a)^3 + \frac{1}{24}\,f''''(a)(x-a)^4 + ... \tag{12}$$

The Solution To Our Hypothetical Problem

Our hypothetical problem equation has terms up to the third order (i.e. x^3). In this case fourth order derivatives and higher are zero and therefore we can ignore them. Our hypothetical problem equation evaluated at the arbitrary point a and the relevant derivatives are...

$$f(a) = 100 - 2\,a + 4\,a^2 - 3\,a^3 \ \Big|\ f'(a) = -2 + 8\,a - 9\,a^2 \ \Big|\ f''(a) = 8 - 18\,a \ \Big|\ f'''(a) = -18 \tag{13}$$

If we set the arbitrary point a to be equal to 2 then we can rewrite Equation (13) above as...

$$f(a) = 88 \ \Big|\ f'(a) = -22 \ \Big|\ f''(a) = -28 \ \Big|\ f'''(a) = -18 \tag{14}$$

Using Equation (14) above and using Equation (12) as our guide then the Taylor Series Expansion for our hypothetical problem is...

$$f(x) = 88 - 22\,(x-a) - \frac{1}{2}\,28\,(x-a)^2 - \frac{1}{6}\,18\,(x-a)^3 \ \text{...where...}\ a = 2 \tag{15}$$

We can rewrite Equation (15) above as...

$$f(x) = 88 - 22\,(x-2) - 14\,(x-2)^2 - 3\,(x-2)^3 \tag{16}$$

We will use Equation (16) above to answer our hypothetical problem questions...

Question 1: What is the value of $f(x)$ when $x = 5$?

Answer: $f(5) = 88 - 22\,(5-2) - 14\,(5-2)^2 - 3\,(5-2)^3 = -185$

Question 2: What is the value of $f(x)$ when $x = -3$?

Answer: $f(-3) = 88 - 22\,(-3-2) - 14\,(-3-2)^2 - 3\,(-3-2)^3 = 223$